Double-critical graph conjecture

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Introduction

Background

Figure: Paul Erdős (left) and Laszlo Lovász (right)

1967 "Theory of Graphs", Erdős.

Introduction

Double-critical graph

A k-chromatic double-critical graph is a connected graph with chromatic number k and such that deleting any pair of adjacent vertices reduces the chromatic number to precisely $k - 2$.

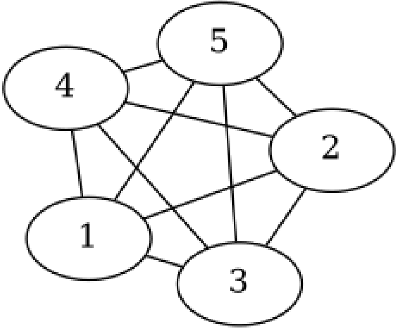

Figure: K_k is a k-chromatic double-critical graph

Other examples?

Conjecture (Erdős 1967)

K_k is the only k-chromatic double critical graph.

Basic intuition: G is k-chromatic double-critical:

- G is connected tightly enough to be k-chromatic
- Every edge is involved in some essential subgraph of G which makes it k-chromatic.
- These subgraphs don't make independent connected components (G is connected).

Conjecture

For $s, t \geq 2$*, and* G *a graph such that* $\omega(G) < \chi(G) = s + t - 1$*, then there exist disjoint subgraphs* G_1, G_2*, such that* $\chi(G_1) \geq s, \chi(G_2) \geq t$*.*

Corollary (and special case)

K_k *is the only* k*-chromatic double critical graph.*

Proof.

Let G be $k+1$-chromatic double-critical. For any $G_1 \leq G$, either $\chi(G_1) < 2$, or $\chi(G_1) \geq 2$, in which case, G_1 must have at least an edge H as a subgraph. But then, $G_1 \cap G_2 = \emptyset$, which means $G_2 \leq G - G_1 \leq G - H$ so $\chi(G_2) \leq \chi(G - H) = k + 1 - 2 < k$. By the contrapositive of the Erdős-Lovász Tihani conjecture for $s = 2$, $\omega(G) = \chi(G) = k + 1$. But then, if we delete any edge from G, we reduce the chromatic number, so all edges of G are in the K_{k+1} subgraph of G, with G connected, $G = K_{k+1}$. □

Research directions

Any first impressions?

Research directions

Theory

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- Limiting k (basically all of the literature).

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- Weakening the conjecture:

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  - Every k-chromatic double-critical graph has a K_k minor.

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- Limiting k (basically all of the literature).
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- General observations on double-critical graphs.

Theory

Forbidden (induced) subgraphs

Disallowing subgraphs from closed neighbourhoods allows us to add edges in the neighbourhoods.

- P_2-free + connected $\implies$ proven and trivial.
- Claw-free (3-star-free) $\implies$ proven until $k \leq 8$.
- ℓ-star free?
- General graphs $\implies$ proven until $k \leq 5$ (but $k \in \{2, 3\}$ are trivial and 4 is easy).

Theory

General observations

Theorem (Huang, Yu)

If K_{k-1} subgraph of G, then $G \cong K_k$.

Theorem (Huang, Yu[2])

If $xy \in E(G)$, then x, y share at least one common neighbour in each color-set.

Corollary

K_3, K_4 are the only double-critical graphs of chromatic numbers 3 and 4 respectively.

[2] Proof given does not require claw-free

Theorem

If G is also claw-free, double-critical, and 5-chromatic, then $G \cong K_5$.

Proof.

Pick adjacent vertices $x, y \in G$, and color $G - x, y$ with 3 colors; common neighbours of x and y are at least 3, one of each color. But x, u_1, u_2, u_3 can't form a claw, so w.l.o.g. $u_1 u_2 \in E(G)$, then: $\{x, u_1, u_2, y\}$ is a K_4 in G, so $G \cong K_5$. □

Take-away intuition

Find cliques in common neighbourhoods. *Eliminating ℓ-stars allows us to add at least one edge in a neighbourhood of size ℓ.*

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Computational approaches

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- Issues

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  - Graph coloring on general graphs is NP-complete.

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  - The search space for a fixed chromatic number is infinite.

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  - DP-like ideas

Computational approaches

Naive approach

Theorem (Pedersen 2012)

There is no non-complete double-critical graph on less than 13 vertices.

Back of the envelope calculation: 13 vertices $\implies \binom{13}{2} = 78$ possible edges $\implies 2^{78}$ labeled graphs $\implies$ running time is unreasonably big.

Computational approaches

Minimum degree

Theorem (Kawarabayashi, Pedersen, Toft 2010)

If G is double-critical, non-complete, and k-chromatic, then $\delta(G) \geq k + 1$.

Example use: for k-chromatic, double-critical graphs with $n = k + 3$ vertices, the search space is reduced to one non-edge per pair of vertices. The search space grows as k, n get bigger and/or farther apart.

Computational approaches

Graphs with claws

An induced claw in a graph determines $\binom{4}{2} = 6$ edges. You can start with a claw and add non-edges (previous theorem) until you get a convenient minimum degree. Reduces search space by up to a factor of 2^6.

Computational approaches

Reduce search space early

Lemma

If G is non-complete, k-chromatic, double critical, then $G - \{x, y\}$ for any adjacent x, y has minimum degree $k - 1$ and is $k - 2$-chromatic.

Idea: generate $G - \{x, y\}$ which satisfies the conditions, then add x, y to generate G which is k-chromatic, then test whether G is double-critical.

Thank you!